Singular vectors of representations of quantum groups

1992 J. Phys. A: Math. Gen. 25 149

(http://iopscience.iop.org/0305-4470/25/1/019)

J. Phys. A: Math. Gen. 25 (1992) 149–160. Printed in the UK

Singular vectors of representations of quantum groups

V K Dobrev†

Institute for Theoretical Physics, University of Göttingen, Bunsenstrasse 9, 3400 Göttingen, Federal Republic of Germany.

Received 25 Jume 1991

Abstract. In the present article we give explicit formulae for singular vectors of Verma modules over $U_q(\mathcal{G})$. We give the general formula for $\mathcal{G} = A_n$, for some cases when $\mathcal{G} = D_n$ and for some rank-two subalgebras of $\mathcal{G} \neq A_n, D_n$. For this we use a special basis of $U_q(\mathcal{G}^-)$, where $\mathcal{G}^-$ is the negative root subalgebra of $\mathcal{G}$, which was introduced in our earlier work on the case $q = 1$. This basis seems more economical than the Poincaré–Birkhoff–Witt type of basis used by Malikov, Feigin and Fuchs for the construction of singular vectors of Verma modules in the case $q = 1$. Furthermore this basis turns out to be part of a general basis introduced recently for other reasons by Lusztig for $U_q(\mathcal{B}^-)$, where $\mathcal{B}^-$ is a Borel subalgebra of $\mathcal{G}$.

1. Introduction

We consider the q-deformation $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ of simple Lie algebras $\mathcal{G}$ which are also called quantum groups [1] or quantum universal enveloping algebras [2, 3]. They arose in the study of the algebraic aspects of quantum integrable systems [4–6]. For recent reviews we refer to [7]. In [6b] for $\mathcal{G} = \mathrm{sl}(2, \mathbb{C})$ and in [1, 8] in general it was observed that the algebras $U_q(\mathcal{G})$ have the structure of a Hopf algebra. This new algebraic structure was further studied in [9–11]. The representations of $U_q(\mathcal{G})$ were considered in [3, 5, 9, 12] for generic values of the deformation parameter. In fact all results from the representation theory of $\mathcal{G}$ carry over to the quantum group case. This is not so, however, if the deformation parameter q is a root of unity. Thus this case is very interesting from the mathematical point of view (see, e.g., [13–15]). Lately, quantum groups were intensively applied (with special emphasis on the case when q is a root of unity) in rational conformal field theories [16–21] and in two-dimensional quantum gravity [22].

In [23] we began the study of the representation theory of $U_q(\mathcal{G})$ when the deformation parameter q is a root of unity. We consider the induced highest weight modules (HWM) over $U_q(\mathcal{G})$, which are also called Verma modules. They all are reducible for $q^N = 1, N \in \mathbb{N} + 1$. In [23] we adapted to $U_q(\mathcal{G})$ the previously developed approach of multiplet classification of Verma modules over (infinite-dimensional) (super-) Lie algebras [24–27]. In [28–30] we gave the character formulae for the irreducible HWM over $U_q(\mathcal{G})$ when $\mathcal{G} = \mathrm{sl}(3, \mathbb{C})$.

† Alexander von Humboldt Foundation fellow; permanent address: Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria.

0305-4470/92/010149+12$03.50

These developments use results on the embeddings of the reducible Verma modules. These embeddings are realized by the so-called *singular* vectors (or *null* or *extremal* vectors). In [23] we gave the general formula for the singular vectors which, however, was not so explicit. Some explicit formulae for singular vectors of Verma modules over $U_q(A_n)$ were presented in [31].

In the present article we give explicit formulae for the singular vectors of Verma modules over $U_q(\mathcal{G})$. We give the general formula for $\mathcal{G} = A_n$, for some cases when $\mathcal{G} = D_n$ and for some rank-two subalgebras of $\mathcal{G} \neq A_n, D_n$. As in [23] and [31] we use a special basis of $U_q(\mathcal{G}^-)$, where $\mathcal{G}^-$ is the negative root subalgebra of $\mathcal{G}$, which was introduced in our earlier work on the case $q = 1$ [24, 32]. This basis seems more economical than the Poincaré–Birkhoff–Witt type of basis used by Malikov, Feigin and Fuchs [33] for the construction of singular vectors of Verma modules in the case $q = 1$. Furthermore our basis turns out to be part of a general basis introduced recently for other reasons by Lusztig [34] for $U_q(\mathcal{B}^-)$, where $\mathcal{B}^-$ is a Borel subalgebra of $\mathcal{G}$.

2. Definitions

Let $\mathcal{G}$ be any complex simple Lie algebra; then $U_q(\mathcal{G})$ is defined [1, 8] as the associative algebra over $\mathbb{C}$ with generators $X_i^\pm, H_i, i = 1, \ldots, l = \operatorname{rank} \mathcal{G}$ and with relationships

$$[H_i, H_j] = 0 \qquad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm \tag{1}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q_i^{H_i/2} - q_i^{-H_i/2}}{q_i^{1/2} - q_i^{-1/2}} = \delta_{ij} [H_i]_{q_i} \qquad q_i = q^{(\alpha_i, \alpha_i)/2} \tag{2}$$

$$\sum_{k=o} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0 \qquad i \neq j \tag{3}$$

where $(a_{ij}) = (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathcal{G}$, $(\cdot, \cdot)$ is the scalar product of the roots normalized so that for the short simple roots α we have $(\alpha, \alpha) = 2$, $n = 1 - a_{ij}$,

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \qquad [m]_q! = [m]_q [m-1]_q \ldots [1]_q \tag{4a}$$

$$[m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} = \frac{\sinh(mh/2)}{\sinh(h/2)} = \frac{\sin(\pi m \tau)}{\sin(\pi \tau)} \qquad q = e^h = e^{2\pi i \tau}, h, \tau \in \mathbb{C} \tag{4b}$$

$$q_i^{a_{ij}} = q^{(\alpha_i, \alpha_j)} = q_j^{a_{ji}}. \tag{4c}$$

This definition is also valid for arbitrary affine Lie algebras [1]. Furthermore we shall omit the subscript q in $[m]_q$ if no confusion can arise. Note also that instead of q some authors use $q' = q^2$. For $q \to 1$, $(h \to 0)$, we recover the commutation relationships from (1) and (2) and Serre's relationships from (3) in terms of the Chevalley generators $H_i, X_i^\pm$. The elements H_i span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, while the elements $X_i^\pm$ generate the subalgebras $\mathcal{G}^\pm$. We shall use the standard decompositions $\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\beta \in \Delta} \mathcal{G}_\beta = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G} \Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}$, Δ^+, Δ^-, the sets of positive, negative, roots, respectively. We recall that H_i correspond to the simple roots α_i of $\mathcal{G}$, and if $\beta = \sum_i n_i\alpha_i$, then $H_\beta = \sum_i n_i H_i$ corresponds to β. The elements of $\mathcal{G}$ which span $\mathcal{G}_\beta$, (recall that dim $\mathcal{G}_\beta = 1$), will be denoted by X_β. These Cartan–Weyl generators are normalized so that

$$[X_\beta, X_{-\beta}] = [H_\beta]_{q_\beta} \qquad \text{for } \beta \in \Delta^+, q_\beta = q^{(\beta,\beta)/2}. \tag{5}$$

In [6b] for $\mathcal{G} = \mathrm{sl}(2,\mathbb{C})$ and in [1, 7] in general it was observed that the algebra $\mathrm{U}_q(\mathcal{G})$ is a Hopf algebra [35]. However, we shall not use this and consequently we shall not introduce the corresponding structure.

3. Highest weight modules over $\mathbf{U_q}(\mathcal{G})$

The HWM V over $\mathrm{U}_q(\mathcal{G})$ [2] are given by their highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V$ such that

$$X_i^+ v_0 = 0 \qquad i = 1, \ldots, l \qquad H v_0 = \lambda(H) v_0 \qquad H \in \mathcal{H}. \tag{6}$$

We start with the *induced* HWM or *Verma* modules V^λ such that $V^\lambda \cong \mathrm{U}_q(\mathcal{G}) \otimes_{\mathrm{U}_q(\mathcal{B})} v_0$ $\cong \mathrm{U}_q(\mathcal{G}^-) \otimes v_0$, where $\mathcal{B} = \mathcal{B}^+$, $\mathcal{B}^\pm = \mathcal{H} \oplus \mathcal{G}^\pm$ are Borel subalgebras of $\mathcal{G}$. (Then the algebras $\mathrm{U}_q(\mathcal{B}^\pm)$ with generators $H_i, X_i^\pm$ are Hopf subalgebras of $\mathrm{U}_q(\mathcal{G})$ [2].) The representation theory of V^λ parallels the theory of Verma modules $V(\Lambda)$ over $\mathcal{G}$. ($V(\Lambda)$ is defined as the HWM over $\mathcal{G}$ induced from the one-dimensional representations of $\mathcal{B}$.) In particular, we shall consider the irreducible HWM L_λ over $\mathrm{U}_q(\mathcal{G})$ as factor modules V^λ/I^λ, where I^λ is the maximal submodule of V^λ.

We recall several facts from [23, 31]. If q is not a root of unity then the Verma module V^λ is reducible iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$[(\lambda + \rho, \beta^\vee) - m]_{q_\beta} = 0 \qquad \beta^\vee \equiv 2\beta/(\beta,\beta) \tag{7}$$

holds, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. (Condition (7) is the generalization of the Verma modules reducibility conditions for finite-dimensional $\mathcal{G}$ [36] and for affine Lie algebras [37].) If (7) holds then there exists a vector $v_s \in V^\lambda$, called a *singular vector*, such that $v_s \neq v_0$, $X_i^+ v_s = 0, i = 1, \ldots, l$, $H v_s = (\lambda(H) - m\beta(H))v_s$, $H \in \mathcal{H}$. The space $U(\mathcal{G}^-)v_s$ is a proper submodule of V^λ isomorphic to the Verma module $V^{\lambda - m\beta} = U(\mathcal{G}^-) \otimes v_0'$ where v_0' is the highest weight vector of $V^{\lambda-m\beta}$; the isomorphism being realized by $v_s \mapsto 1 \otimes v_0'$. The singular vector is given by [24, 25, 23]

$$v_s = v^{\beta,m} = \mathcal{P}_m^\beta(X_1^-, \ldots, X_l^-) \otimes v_0 \tag{8}$$

where $\mathcal{P}_m^\beta$ is a homogeneous polynomial in its variables of degrees mn_i, where $n_i \in \mathbb{Z}_+$ come from $\beta = \sum n_i\alpha_i$, where α_i is the system of simple roots. The polynomial $\mathcal{P}_m^\beta$ is unique up to a non-zero multiplicative constant.

The Verma module V^λ contains a unique proper maximal submodule I^λ. Among the HWM with highest weight λ there is a unique irreducible one, denoted by L_λ, i.e.

$$L_\lambda = V^\lambda/I^\lambda. \tag{9}$$

If V^λ is irreducible then $L_\lambda = V^\lambda$. Thus we discuss L_λ for which V^λ is reducible. Consider V^λ reducible with respect to every simple root (and thus to all positive roots):

$$[(\lambda+\rho,\alpha_i^\vee) - m_i]_{q_i} = [(\lambda)(H_i) + 1 - m_i]_{q_i} = 0 \qquad i = 1,\ldots,l \tag{10}$$

where we have used $\rho(\alpha_i^\vee) = 1$. Then L_λ is a finite-dimensional highest weight module over $\mathcal{G}$ [3, 5, 10]. If we restrict $\mathcal{G}$ to its compact real form $\mathcal{G}_c$ then the set of all L_λ coincides with the set of all finite dimensional unitary irreducible representations of $\mathcal{G}_c$. (In the case of affine Lie algebras, L_λ with (10) holding are the so-called integrable HWM [38].) An important class of the case when (10) holds are the so-called fundamental representations L_{λ_i}, $i = 1,\ldots,l$ characterized by $(\lambda_i,\alpha_j^\vee) = \delta_{ij}$, i.e. $(\lambda_i+\rho,\alpha_j^\vee) = 1+\delta_{ij} = m_j(\lambda_i)$. The representations of $\mathrm{U}_q(\mathcal{G})$ are deformations of the representations of $U(\mathcal{G})$, and the latter are obtained from the former for $q \to 1$ [10].

Recently, De Concini and Kac [13] have given a formula for the determinant of the contravariant form on the Verma modules V^λ. This result implies, in the usual way, the description of irreducible subquotients of V^λ. In particular, this confirms our results on the embeddings of the reducible modules V^λ [23], part of which were summarized earlier.

4. Singular vectors for generic q

4.1. The case of the simple roots

Let $\beta = \alpha_j$; then from expression (8) we have

$$v^{j,m} = (X_j^-)^m \otimes v_0. \tag{11}$$

We obtain, using (2),

$$[X_j^+,(X_j^-)^m] = \sum_{k=0}^{m-1} (X_j^-)^{m-1-k}[H_j]_{q_j}(X_j^-)^k$$

$$= (X_j^-)^{m-1}\sum_{k=0}^{m-1}[H_j-2k]_{q_j} = (X_j^-)^{m-1}[m]_{q_j}[H_j-m+1]_{q_j}. \tag{12}$$

If $v^{j,m}$ is a singular vector we should have

$$0 = X_j^+ v^{j,m} = [X_j^+,(X_j^-)^m]\otimes v_0 = (X_j^-)^{m-1}[m]_{q_j}[\lambda(H_j)-m+1]_{q_j}\otimes v_0. \tag{13}$$

(Note that $X_k^+ v^{j,m} = 0$, for $k \neq j$.) If $q_j = q^{(\alpha_j,\alpha_j)/2}$ is not a root of unity (13) gives just condition (7).)

4.2.

As another example we take a root β which is the sum of two simple roots of equal length: $\beta = \alpha_1 + \alpha_2$, $(\beta, \beta) = (\alpha_j, \alpha_j)$, $j = 1, 2$, $\beta^\vee = \alpha_1^\vee + \alpha_2^\vee$. This case is relevant for $U_q(\mathcal{G})$ for $\mathcal{G} = A_n, n > 1$, $\mathcal{G} = B_n, n > 2$, $\mathcal{G} = C_n, n > 2$, $\mathcal{G} = D_n, n > 3$, $\mathcal{G} = E_6, E_7, E_8, F_4$. For $\mathcal{G} = B_n$ the two roots α_1, α_2 are long, for $\mathcal{G} = C_n$ they are short, while for $\mathcal{G} = F_4$ there is one case when they are long and one case when they are short. Let us have condition (7) fulfilled for β, but not for $\alpha_j, j = 1, 2$:

$$[(\lambda + \rho, \beta^\vee) - m]_{q_\beta} = [\lambda(H_\beta) + 2 - m]_{q_\beta} = 0 \qquad q_\beta = q_1 = q_2 \tag{14a}$$

$$[(\lambda + \rho, \alpha_j^\vee) - m']_{q_j} \neq 0 \qquad j = 1, 2 \qquad \forall m' \in \mathbb{Z}_+. \tag{14b}$$

(The reason for the appearance of $\mathbb{Z}_+$ in (14*b*) instead of $\mathbb{N}$ will become clear in subsection 4.6.) Then one can check that the singular vector is given by [23]

$$v^{\beta,m} = \sum_{k=0}^{m} c^1_{mk} (X_1^-)^{m-k} (X_2^-)^m (X_1^-)^k \otimes v_0 \tag{15a}$$

$$= \sum_{k=0}^{m} c^2_{mk} (X_2^-)^{m-k} (X_1^-)^m (X_2^-)^k \otimes v_0 \tag{15b}$$

$$c^i_{mk} = (-1)^k c^i \binom{m}{k}_{q_i} \frac{[\lambda(H_i) + 1]_{q_i}}{[\lambda(H_i) + 1 - k]_{q_i}} \qquad i = 1, 2 \qquad c^i \neq 0. \tag{15c}$$

For this check we also need the following formula involving the q-hypergeometric function ${}_2F_1^q$:

$${}_2F_1^q(-k, s; s + 1 - p; q^{(p-k)/2}) = \delta_{p0} \frac{[k]![s]!}{[k + s]!} q^{ks/2} \tag{16a}$$

where

$${}_2F_1^q(a, b; c; z) \equiv \sum_{n \in \mathbb{Z}_+} \frac{[a+n]![b+n]![c]!}{[a]![b]![c+n]![n]!} z^n. \tag{16b}$$

Such special q-functions are discussed in [3, 39].

For $q \to 1$ formula (15) goes to the correct formula in the same situation [24–32] (cf formulae (8.40) and (8.41)). One should notice that this is not the ordered Poincaré–Birkhoff–Witt type of basis. This basis involves only simple root space vectors and it was used in our earlier work in the case $q = 1$ [24–32]. We think that this basis is more economical for the construction of singular vectors. It is very interesting that our basis turns out to be part of a general basis introduced recently for other reasons by Lusztig [34] for $U_q(\mathcal{B}^-)$.

4.3.

Let $\beta = \alpha_1 + \alpha_2$, where α_1 is a long simple root and α_2 a short simple root (cf (17a)) so that

$$a_{12} = -1 \qquad a_{21} = -1 - \varepsilon \qquad \varepsilon = 1, 2 \qquad q_1 = q^{1+\varepsilon} \qquad q_2 = q \tag{17a}$$

$$\beta^\vee = \beta = (1+\varepsilon)\alpha_1^\vee + \alpha_2^\vee \qquad q_\beta = q \tag{17b}$$

and let

$$[(\lambda+\rho, \beta^\vee) - m]_{q_\beta} = [\lambda(H_1)(1+\varepsilon) + \lambda(H_2) + 2 + \varepsilon - m]_{q_\beta} = 0 \tag{18a}$$

$$[(\lambda+\rho, \alpha_j^\vee) - m']_{q_j} \neq 0 \qquad j = 1, 2, \forall m' \in \mathbb{Z}_+. \tag{18b}$$

The case $\varepsilon = 1$ is relevant for $\mathcal{G} = B_n, C_n, F_4$, while the case $\varepsilon = 2$ is relevant for $\mathcal{G} = G_2$. Now one can check that the singular vector is given by

$$v^{\beta,m} = \sum_{k=0}^{m} c^{11}_{mk} (X_1^-)^{m-k} (X_2^-)^m (X_1^-)^k \otimes v_0 \tag{19a}$$

$$c^{11}_{mk} = (-1)^k c^{11} \binom{m}{k}_{q_1} \frac{[\lambda(H_1)+1]_{q_1}}{[\lambda(H_1)+1-k]_{q_1}}. \tag{19b}$$

4.4.

Let $\beta = \alpha_1 + 2\alpha_2$, where α_1, α_2 are as in the previous case (a long and a short simple root), cf (17a), so that

$$\beta^\vee = \varepsilon\beta/2 = \varepsilon(1+\varepsilon)\alpha_1^\vee/2 + \varepsilon\alpha_2^\vee \qquad q_\beta = q^{2/\varepsilon} \tag{20}$$

and let

$$\begin{aligned}[(\lambda+\rho, \beta^\vee) - m]_{q_\beta} &= [\varepsilon(1+\varepsilon)\lambda(H_1)/2 + \varepsilon\lambda(H_2) + \varepsilon(3+\varepsilon)/2 - m]_{q_\beta} \\ &= [(1+\varepsilon)\lambda(H_1) + 2\lambda(H_2) + (3+\varepsilon) - 2m/\varepsilon]_q = 0 \end{aligned} \tag{21a}$$

$$[(\lambda+\rho, \alpha_j^\vee) - m']_{q_j} \neq 0 \qquad j = 1, 2 \qquad \forall m' \in \mathbb{Z}_+. \tag{21b}$$

Now one can check that the singular vector for $\varepsilon = 1$ is given by

$$v^{\beta,m}_{\varepsilon=1} = \sum_{k=0}^{2m} c^{21}_{mk} (X_2^-)^{2m-k} (X_1^-)^m (X_2^-)^k \otimes v_0 \tag{22a}$$

$$c^{21}_{mk} = (-1)^k c^{21} \binom{2m}{k}_q \frac{[\lambda(H_2)+1]_q}{[\lambda(H_2)+1-k]_q}. \tag{22b}$$

4.5.

Let $\mathcal{G} = G_2$, let $\beta = \alpha_1 + 3\alpha_2$, where α_1 is the long and α_2 the short simple root, cf (17*a*), so that

$$\beta^\vee = \beta/3 = \alpha_1^\vee + \alpha_2^\vee \qquad q_\beta = q^3 \tag{23}$$

and let

$$[(\lambda+\rho,\beta^\vee) - m]_{q_\beta} = [\lambda(H_1) + \lambda(H_2) + 2 - m]_{q_\beta} = 0 \tag{24a}$$

$$[(\lambda+\rho,\alpha_j^\vee) - m']_{q_j} \neq 0 \qquad j = 1,2 \qquad \forall m' \in \mathbb{Z}_+. \tag{24b}$$

Now one can check that the singular vector is give

$$v^{\beta,m} = \sum_{k=0}^{3m} c_{mk}^{31} (X_2^-)^{3m-k} (X_1^-)^m (X_2^-)^k \otimes v_0 \tag{25a}$$

$$c_{mk}^{31} = (-1)^k c^{31} \binom{3m}{k}_{q^3} \frac{[\lambda(H_2)+1]_{q^3}}{[\lambda(H_2)+1-k]_{q^3}}. \tag{25b}$$

4.6.

Let $\mathcal{G} = A_l$ and let $\alpha_i, i = 1,\ldots,l$ be the simple roots, so that $(\alpha_j,\alpha_k) = -1$ for $|j-k| = 1$ and $(\alpha_j,\alpha_k) = 2\delta_{jk}$ for $|j-k| \neq 1$. Then every root $\beta \in \Delta^+$ is given by $\beta = \beta_{in} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+n-1}$, where $1 \leqslant i \leqslant l$, $1 \leqslant n \leqslant l-i+1$. Recall that a root $\tilde\alpha \in \Delta^+$ is called the *highest root* of Δ if $\tilde\alpha + \beta$ is not a root for any $\beta \in \Delta^+$. For A_l the highest root is given by $\tilde\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. Thus every root $\beta \in \Delta^+$ is the highest root of a subalgebra of A_l; explicitly β_{in} is the highest root of the subalgebra A_n with simple roots $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+n-1}$. This means that it is sufficient to give the formula for the singular vector corresponding to the highest root.

Let us have condition (7) fulfilled for $\tilde\alpha$, but not for any other positive root:

$$[(\lambda+\rho,\tilde\alpha^\vee) - m]_q = [\lambda(H_{\tilde\alpha}) + l - m]_q = 0 \tag{26a}$$

$$[(\lambda+\rho,\beta_{in}^\vee) - m']_q = [\lambda(H_{in}) + n - m']_q \neq 0 \qquad n \neq l \qquad \forall m' \in \mathbb{Z}_+. \tag{26b}$$

Now one can check that the angular vector is given by

$$\begin{aligned} v^{\tilde\alpha,m} = \sum_{k_1=0}^{m} \cdots \sum_{k_{l-1}=0}^{m} & c_{k_1,\ldots,k_{l-1}} (X_1^-)^{m-k_1} \ldots (X_{l-1}^-)^{m-k_{l-1}} \\ & \times (X_l^-)^m (X_{l-1}^-)^{k_{l-1}} \ldots (X_1^-)^{k_1} \otimes v_0 \end{aligned} \tag{27a}$$

$$\begin{aligned} c_{k_1,\ldots,k_{l-1}} = (-1)^{k_1+\cdots+k_{l-1}} & c^l \binom{m}{k_1}_q \cdots \binom{m}{k_{l-1}}_q \\ & \times \frac{[(\lambda+\rho)(H^1)]}{[(\lambda+\rho)(H^1) - k_1]} \cdots \frac{[(\lambda+\rho)(H^{l-1})]}{[(\lambda+\rho)(H^{l-1}) - k_{l-1}]} \qquad c^l \neq 0 \end{aligned} \tag{27b}$$

where $H^s = H_{\beta_{1s}} = H_1 + H_2 + \cdots + H_s$. Note that for $l = 2$ formula (27) coincide with formula (15) with $q_j = q$. Formula (27) for $l = 3$ may be written equivalently as

$$v^{\tilde{\alpha},m} = \sum_{k_1=0}^{m} \sum_{k_2=0}^{m} c'_{k_1,k_2} (X_1^-)^{m-k_1} (X_3^-)^{m-k_2} (X_2^-)^m (X_3^-)^{k_2} (X_1^-)^{k_1} \otimes v_0 \tag{28a}$$

$$c'_{k_1,k_2} = (-1)^{k_1+k_2} c'^2 \binom{m}{k_1}_q \binom{m}{k_2}_q \frac{[(\lambda+\rho)(H_1)]}{[(\lambda+\rho)(H_1) - k_1]} \frac{[(\lambda+\rho)(H_3)]}{[(\lambda+\rho)(H_3) - k_2]} \tag{28b}$$

and for $q \to 1$ gives the correct formula in the same situation [24, 32] (cf formula (8.42)).

4.7.

Let $\mathcal{G} = D_l$, $l \geqslant 4$, and let α_i, $i = 1, \ldots, l$ be the simple roots, so that

$$(\alpha_i, \alpha_j) = \begin{cases} -1 & |i-j| = 1, i, j \neq l \\ -1 & ij = l(l-2) \\ 2 & i = j \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Let us consider roots $\beta_i \in \Delta^+$ given by $\beta_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_l$, where $1 \leqslant i \leqslant l-3$. Note that β_i is a root of the subalgebra D_{l-i+1} with simple roots $\alpha_i, \alpha_{i+1}, \ldots, \alpha_l$. This means that, in order to account for all roots β_i, it is sufficient to give the formula for the singular vector corresponding to the root $\tilde{\beta} = \beta_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. (This is not the highest root of D_l.)

Let us have condition (7) fulfilled for $\tilde{\beta}$, but not for any subroot γ of $\tilde{\beta}$ ($\gamma' \in \Delta^+$ is a subroot of $\gamma'' \in \Delta^+$ if $\gamma'' - \gamma'$ may be expressed as a linear combination of simple roots with non-negative coefficients):

$$[(\lambda + \rho, \tilde{\beta}^\vee) - m]_q = [\lambda(H_{\tilde{\beta}}) + l - m]_q = 0. \tag{30}$$

Now one can check that the singular vector is given by

$$v^{\tilde{\beta},m} = \sum_{k_1=0}^{m} \cdots \sum_{k_{l-1}=0}^{m} \tilde{c}_{k_1,\ldots,k_{l-1}} (X_1^-)^{m-k_1} \cdots (X_{l-3}^-)^{m-k_{l-3}} (X_{l-1}^-)^{m-k_{l-1}}$$
$$\times (X_l^-)^{m-k_{l-2}} (X_{l-2}^-)^m (X_l^-)^{k_{l-2}} (X_{l-1}^-)^{k_{l-1}} (X_{l-3}^-)^{k_{l-3}} \cdots (X_1^-)^{k_1} \otimes v_0 \tag{31a}$$

$$\tilde{c}_{k_1,\ldots,k_{l-1}} = (-1)^{k_1+\cdots+k_{l-1}} \tilde{c}^l \binom{m}{k_1}_q \cdots \binom{m}{k_{l-1}}_q$$
$$\times \frac{[(\lambda+\rho)(H^1)]}{[(\lambda+\rho)(H^1) - k_1]} \cdots \frac{[(\lambda+\rho)(H^{l-3})]}{[(\lambda+\rho)(H^{l-3}) - k_{l-3}]}$$
$$\times \frac{[(\lambda+\rho)(H_{l-1})]}{[(\lambda+\rho)(H_{l-1}) - k_{l-1}]} \frac{[(\lambda+\rho)(H_l)]}{[(\lambda+\rho)(H_l) - k_{l-2}]} \qquad \tilde{c}^l \neq 0 \tag{31b}$$

where $H^s = H_{\beta_s} = H_1 + H_2 + \cdots + H_s$. Note that if we set formally $l = 3$ in these formulae they will coincide with the formulae for $A_3 \cong D_3$, in particular in the form (28), identifying the roots $(\alpha_1, \alpha_2, \alpha_3)_{D_3} \to (\alpha_2, \alpha_1, \alpha_3)_{A_3}$.

4.8.

The singular vectors given in (15), (19), (22), (25), (27), (28) and (31) are in the generic situation, i.e. when condition (7) is fulfilled for β, but not for the subroots of β. Let us consider formula (15) or (27) for $l = 2$ when

$$[(\lambda+\rho,\alpha_j^\vee)-m_j]_{q_j}=0 \qquad j=1,2 \qquad m_j\in\mathbb{Z}_+ \qquad m_1+m_2\in\mathbb{N} \tag{32}$$

i.e. condition (14) is fulfilled in addition for at least one of the roots α_1, α_2, and for the other root it may be broken only in the sense that the corresponding number m_k may be equal to zero. Then formulae (15a) and (15b) reduce to

$$v^{\beta,m} = c_1(X_1^-)^{m_2}(X_2^-)^m(X_1^-)^{m_1}\otimes v_0 \tag{33a}$$
$$= c_2(X_2^-)^{m_1}(X_1^-)^m(X_2^-)^{m_2}\otimes v_0 \tag{33b}$$
$$= (X_2^-)^{m_1}\sum_{k=0}^{m_2} c^1_{m_2k}(X_1^-)^{m_2-k}(X_2^-)^{m_2}(X_1^-)^{k+m_1}\otimes v_0 \tag{33c}$$
$$= (X_1^-)^{m_2}\sum_{k=0}^{m_1} c^2_{m_1k}(X_2^-)^{m_1-k}(X_1^-)^{m_1}(X_2^-)^{k+m_2}\otimes v_0 \tag{33d}$$

where $m = m_1 + m_2 \in \mathbb{N}$, $c^1_{m_2k}$, $c^2_{m_1k}$, respectively, is given by (15c) with λ replaced by the Weyl dot reflection shifted highest weight $\lambda - m\alpha_1 = s_1\cdot\alpha_1$, $\lambda - m\alpha_2 = s_2\cdot\alpha_2$, respectively, i.e. with $\lambda(H_i)+1$ replaced by $-m_i$, $i = 1, 2$, respectively. [Weyl dot reflections $w\cdot\lambda$ are defined through the ordinary Weyl reflections $w(\lambda)$ by $w\cdot\lambda \equiv w(\lambda+\rho)-\rho$, where $w \in W$, W is the Weyl group of $\mathcal{G}$ generated by the reflections s_i corresponding to the simple roots α_i, the ordinary Weyl reflections being defined by $s_\alpha(\lambda) \equiv \lambda - (\lambda,\alpha^\vee)\alpha$, for any $\alpha \in \Delta$.] The four expressions in (33) are used to prove commutativity of certain embedding diagrams, in particular the hexagon diagram of $U_q(\mathrm{sl}(3,\mathbb{C}))$ [23] (or, for $q = 1$, the hexagon diagram of $\mathrm{sl}(3,\mathbb{C})$ [25]).

If (32) holds then formula (19a) reduces to

$$v^{\beta,m} = c_1'(X_1^-)^{m-m_1}(X_2^-)^m(X_1^-)^{m_1}\otimes v_0 \qquad m=(1+\varepsilon)m_1+m_2 \tag{34a}$$

formula (22a) reduces to

$$v^{\beta,m}_{\varepsilon=1} = c_2'(X_2^-)^{2m-m_2}(X_1^-)^m(X_2^-)^{m_2}\otimes v_0 \qquad m=m_1+m_2 \tag{34b}$$

formula (25a) reduces to

$$v^{\beta,m} = c_3'(X_2^-)^{3m-m_2}(X_1^-)^m(X_2^-)^{m_2}\otimes v_0 \qquad m=m_1+m_2. \tag{34c}$$

Analogously let us consider formula (28) or (27) for $l = 3$ in the case when condition (7) is also fulfilled for at least one of the simple roots $\alpha_1, \alpha_2, \alpha_3$, i.e.

$$[(\lambda+\rho,\alpha_j^\vee)-m_j]_{q_j}=0 \qquad j=1,2,3 \qquad m_j\in\mathbb{Z}_+ \qquad m=m_1+m_2+m_3\in\mathbb{N}. \tag{35}$$

Denoting $m_{ij} = m_i + m_j$ we write down the reduction of formula (27a) or (28a):

$$v^{\beta,m} = c_1'(X_1^-)^{m_{23}}(X_2^-)^{m_3}(X_3^-)^m(X_2^-)^{m_{12}}(X_1^-)^{m_1}\otimes v_0 \tag{36a}$$
$$= c_2'(X_1^-)^{m_{23}}(X_3^-)^{m_{12}}(X_2^-)^m(X_3^-)^{m_3}(X_1^-)^{m_1}\otimes v_0 \tag{36b}$$
$$= c_3'(X_3^-)^{m_{12}}(X_2^-)^{m_1}(X_1^-)^m(X_2^-)^{m_{23}}(X_3^-)^{m_3}\otimes v_0 \tag{36c}$$

and several other expressions which analogously to (33c) and (33d) use the polynomials corresponding to roots which are the sum of two simple roots (and some expressions which use the trivial commutativity $[X_1^-, X_3^-] = 0$).

5. Singular vectors for q a root of unity

Let $\mathcal{G}$ be an arbitrary simple complex Lie algebra again. Let q be a root of unity. Then all Verma modules V^λ are reducible. For each V^λ there exist singular vectors for arbitrary $\lambda \in \mathcal{H}^*$. They are given explicitly by [23]

$$v^{k_1,\ldots,k_l} = \prod_{j=1}^{l}(X_j^-)^{k_jN_j} \otimes v_0 \qquad k_j \in \mathbb{Z}_+, \sum_{j=1}^{l} k_j > 0 \tag{37}$$

where $N_j \in \mathbb{N}+1$ are the smallest integers such that $q_j^{N_j} = 1$, $j = 1, \ldots, l$. The factors $(X_j^-)^{k_jN_j}$ up to a sign belong to the centre of $\mathrm{U}_q(\mathcal{G})$ [23]. Namely, let $\alpha_i, \alpha_j, i \neq j$ be two simple roots with equal length so that $a_{ij} \neq 0$. Then using Serre relationships (3) and $q_i = q_j$ we obtain

$$X_i^-(X_j^-)^k = -[k-1]_{q_j}(X_j^-)^k X_i^- + [k]_{q_j}(X_j^-)^{k-1}X_i^-X_j^- . \tag{38}$$

Thus if $q_j = \mathrm{e}^{2\pi\mathrm{i}/N_j}$ we have

$$X_i^-(X_j^-)^{k_jN_j} = (-1)^{k_j}(X_j^-)^{k_jN_j}X_i^- . \tag{39}$$

In particular, the elements $(X_j^-)^{2N_j}$ belong to the centre of $\mathrm{U}_q(\mathcal{G})$. It is clear that the Verma submodules of V^λ corresponding to the singular vectors in (37) are explicitly given by $V^{\lambda'}$ with $\lambda' = \lambda - \sum_{j=1}^{l} k_jN_j\alpha_j$.

Besides this there exist other singular vectors if the highest weight λ obeys the condition (7). Consider $\beta \in \Delta^+$, $\beta = \sum n_j\alpha_j$, and let $N_\beta \in \mathbb{N}+1$ be the smallest integer such that $q_\beta^{N_\beta} = 1$, with q_β as in (5). Let us have condition (7) fulfilled for β with some $m \in \mathbb{N}$ but not fulfilled for any subroots of β. Let $k, n \in \mathbb{Z}_+$, $k + n > 0$, $n < N_\beta$ be such that $m = kN_\beta + n$. Then we have the following expression for the singular vector:

$$v^{\beta,n,k} = (\mathcal{P}_n^\beta \mathcal{P}_{N_\beta - n}^\beta)^k \mathcal{P}_n^\beta \otimes v_0 \tag{40}$$

where $\mathcal{P}_u^\beta(X_1^-, \ldots, X_l^-)$ is a homogeneous polynomial as in (8). For explicit expressions of $\mathcal{P}_n^\beta$ we refer to formulae (11), (15), (19), (22), (25), (27), (28), (31), with m replaced by u. It is clear that the submodules of V^λ corresponding to the singular vectors in (40) are explicitly given by $V^{\lambda'}$ with $\lambda' = \lambda - \sum_{j=1}^{l}(k_jN_j + nn_j)\alpha_n j$.

In summary, the singular vectors for q a root of unity which are given by (40) are obtained by combining the factors $\prod_{j=1}^{l}(X_j^-)^{k_jN_j}$ (from (37)) with the polynomials $\mathcal{P}_m^\beta$ (from (8)) giving the singular vectors in the generic case, however with the degree m restricted by N_β.

Acknowledgments

The author would like to thank T Eguchi, M Jimbo, V G Kac, P P Kulish, G Lusztig and N Yu Reshetikhin for discussions. This work was partially supported by the

Ministry of Science of Bulgaria, Grants 3 and 403, and by the Alexander von Humboldt Foundation.

References

[1] Drinfeld V G 1985 *Dokl. Akad. Nauk SSSR* **283** 1060–4; (Engl. transl. 1985 *Sov. Math. Dokl.* **32** 254–8); 1986 *Proc. ICM* (MRSI: Berkeley)
[2] Reshetikhin N Yu 1987 Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links I, II. *Preprint* LOMI Leningrad E-4-87, E-17-87
[3] Kirillov A N and Reshetikhin N Yu 1988 Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links *Preprint* LOMI Leningrad E-9-88
[4] Faddeev L D 1984 Integrable models in 1 + 1 dimensional quantum field theory *Les Houches Lectures 1982* (Amsterdam: Elsevier)
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* vol 151, (Berlin: Springer) pp 61–119
[5] Kulish P P and Reshetikhin N Yu 1981 *Zap. Nauch. Semin. LOMI* **101** 101–10; (Engl. transl. 1983 *J. Sov. Math.* **23** 2435–41)
[6a] Sklyanin E K 1982 *Funkts. Anal. Prilozh.* **16** 27–34 (Engl. Transl. 1982 *Funct. Anal. Appl.* **16** 263–70); 1983 *Funkts. Anal. Prilozh.* **17** 34–48 (Engl. Transl. 1982 *Funct. Anal. Appl.* **17** 274–88)
[6b] Sklyanin E K 1985 *Usp. Mat. Nauk* **40** 214 (in Russian)
[7] Faddeev L D and Takhtajan L A 1985 *Preprint* LPT Paris-VII; 1986 *Lecture Notes in Physics* *276* (Berlin: Springer) pp 166–79
Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1988 *Algebraic Analysis* vol 1 (New York: Academic) p 129; 1989 *Alg. Anal.* **1** 178–206
Jimbo M 1989 *Int. J. Mod. Phys.* A **4** 3759–77
Takhtajan L A 1989 *Adv. Stud. Pure. Math.* **19** 435–57
de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371–465
[8] Jimbo M 1985 *Lett. Math. Phys.* **10** 63–9; 1986 *Lett. Math. Phys.* **11** 247–52
[9] Rosso M 1987 *C. R. Acad. Sci. Paris Serie I* **305** 587–90
Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613–65
[10] Rosso 1987 *Commun. Math. Phys.* **117** 581–93
Lusztig G 1988 *Adv. Math.* **70** 237–49
[11] Verdier J-L 1987 *Séminaire Bourbaki No 685 Astérisque* **152–3** 305
[12] Reshetikhin N Yu and Semenov-Tian-Shansky M A 1990 *Lett. Math. Phys.* **19** 133–42
[13] De Concini C and Kac V G 1990 Representations of quantum groups at roots of 1 *Preprint* Pisa Scuola Normale Superiore math No 75
[14] Date E, Jimbo M, Miki K and Miwa T 1990 R-matrix for cyclic representations of $U_q(sl(3,\mathbb{C})^{(1)})$ at $q^3 = 1$ *Preprint* Kyoto RIMS-696; 1990 Cyclic representations of $U_q(sl(n+1,\mathbb{C}))$ at $q^N = 1$ *Preprint* Kyoto RIMS-703
[15] Lusztig G 1988 *Contemp. Math.* **82** 59–77; 1990 *J. Algebra* **131** 466–75
[16] Alvarez-Gaumé L, Gómez C and Sierra G 1989 *Phys. Lett.* **220B** 142–52; 1989 *Nucl. Phys.* B **319** 155–86; 1990 *Nucl. Phys.* B **330** 347; 1989 *Preprint* CERN-TH.5540/89 to appear in the Memorial Volume of Vadim Knizhnik ed L Brink, D Friedan and A M Polyakov (Singapore: World Scientific)
[17] Moore G and Reshetikhin N Yu 1989 *Nucl. Phys.* B **328** 557–74
Moore G and Seiberg N 1989 *Commun. Math. Phys.* **123** 177–254
[18] Witten E 1990 *Nucl. Phys.* B **330** 285
[19] Ganchev A Ch 1989 *Phys. Lett.* **233B** 374–82
Furlan P, Ganchev A Ch and Petkova V B 1990 *Preprint* INFN Trieste INFN/AE-89/15
[20] Gómez C and Sierra G 1990 *Phys. Lett.* **240B** 149–57; 1990 The quantum symmetry of rational conformal field theories *Preprint* Geneva University, Department of Theoretical Physics
[21] Reshetikhin N Yu and Smirnov F A 1990 *Commun. Math. Phys.* **131** 157–77
[22] Gervais J-L 1990 *Commun. Math. Phys.* **130** 257–83; 1990 *Phys. Lett.* **243B** 85–92; 1990 *Preprint* LPTENS 90/13
[23] Dobrev V K 1989 *ICTP Trieste Internal Report* IC/89/142 to appear in the *Proc. Int. Group Theory Conference (St. Andrews July/August 1989) London Math. Soc. Lecture Note Series*

(Cambridge: Cambridge University Press)

[24] Dobrev V K 1985 *J. Math. Phys.* **26** 235–51; 1985 *Lett. Math. Phys.*9 205

[25] Dobrev V K 1985 *Conf. on Algebraic Geometry and Integrable Systems (Oberwolfach July 1984) Preprint* ICTP Trieste IC/85/9

[26] Dobrev V K 1986 *Proc. XIII Int. Conf. on Differential-Geometric Methods in Theoretical Physics (Shumen, 1984)* ed H D Doebner and T D Palev (Singapore: World Scientific) pp 348–70

[27] Dobrev V K 1986 *Lett. Math. Phys.* **11** 225–34

[28] Dobrev V K 1989 *Proc. 13th Johns Hopkins Workshop on Knots, Topology and Field Theory, (Florence, 1989)* ed L Lusanna (Singapore: World Scientific) pp 539–47

[29] Dobrev V K 1990 *Proc. Quantum Groups Workshop, (Clausthal, 1989) 370* ed H-D Doebner and J-D Hennig *Lecture Notes in Physics* (Berlin: Springer) pp 107–17

[30] Dobrev V K 199X *Int. Coll. Group Theoretical Methods in Physics (Moscow, June 1990)* to appear

[31] Dobrev V K 199X Lecture at the Yukawa International Seminar (Kyoto, May 1990) *Proc. Prog. Theor. Phys. Suppl.* **102** 137–58

[32] Dobrev V K 1988 *Rep. Math. Phys.* **25** 159–81

[33] Malikov F G, Feigin B L and Fuchs D B 1986 *Funkts. Anal. Prilozh.* **20** (2) 25–37 (Engl. transl. *Funct. Anal. Appl.* **20** 103–13)

[34] Lusztig G 1990 Lectures at the 1990 Yukawa International Seminar (Kyoto) *Prog. Theor. Phys. Suppl.* **102** 175–201

[35] Abe E 1980 *Hopf Agebras (Cambridge Tracts in Mathematics 74)* (Cambridge: Cambridge University Press)

[36] Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *Funkts. Amal. Prilozh.* **5** N1 1–9; 1971 (Engl. transl. *Funkt. Anal. Appl.* **5** 1–8)

[37] Kac V G and Kazhdan D 1979 *Adv. Math.* **34** 97-108

[38] Kac V G 1983 *Infinite-dimensional Lie Algebras. An Introduction (Progress in Mathematics 44)* (Boston: Birkhäuser)

[39] Masuda T, Mimachi K, Nakagami Y, Noumi M and Ueno K 1988 *C. R. Acad. Sci. Paris Serie I* **307** 559–64

Koornwinder T H 1989 *Nederrl. Akad. Wetensch. Proc.* A **92** 97–117

Nuomi M and Mimachi K 1990 *Commun. Math. Phys.* **128** 521–31

Klymik A U and Groza V A 1989 *Preprint* Kiev ITP-89-37P (in Russian)

Klymik A U 1990 Talk at the International Colloquium on Group Theoretical Methods in Physics (Moscow June 1990)